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LETTER TO THE EDITOR

Random-field Ising model on complete graphs and trees

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Abstract
We present exact results for the critical behaviour of the random field Ising model on complete graphs and trees, both at equilibrium and away from equilibrium, i.e. models for hysteresis and Barkhausen noise. We show that for stretched exponential and power law distributions of random fields the behaviour on complete graphs is non-universal, while the behaviour on Cayley trees is universal even in the limit of large co-ordination.

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The central issue in the equilibrium random field Ising model (RFIM) is the nature of the phase transition from the ferromagnetic state at weak disorder to the frozen paramagnetic state at high disorder. This issue is important for two reasons. Firstly, from a general theoretical perspective, we need to know if the concept of universality extends to systems which have disorder as well as thermal fluctuations. Secondly it has been claimed that an experimental system which is heavily studied, the diluted antiferromagnet in a field, is in the same universality class as the RFIM [1]. If the RFIM is non-universal, then this correspondence needs to be reconsidered. We do not consider this correspondence directly here, as we concentrate on the general issue of universality in the RFIM.

After some controversy it was rigorously demonstrated that the RFIM transition occurs at a finite width of the distribution in three dimensions [2] and at an infinitesimal width in one and two dimensions. Moreover, Aharony [3] showed that within mean-field theory at low temperatures, the transition is first order for bimodal disorder distributions but second order for unimodal distributions. Numerical studies at zero temperature suggest that in four dimensions the bimodal case is first order and the Gaussian case is second order. The analysis in three dimensions is less conclusive [4]. The difference between the Gaussian and bimodal cases has been attributed to percolative effects [5]. We have recently shown that at zero temperature the mean-field theory is *non-universal* [6] in the sense that the order parameter exponent varies continuously with the disorder, for the distribution function of equation (3) below. Exact optimization calculations [7, 8] in three dimensions have also suggested that the correlation length exponent, as deduced from finite size scaling, is non-universal [9].

Motivated by the fact that the RFIM is non-universal within mean-field theory for the stretched exponential distribution, we have analysed the RFIM on complete graphs with disorder distribution $(\delta h/|h|)^x$ $(0 < x < 1, |h| < \delta h)$. We find that this distribution is anomalous in the sense that this sort of disorder never destroys the spontaneously magnetized state, at least within mean-field theory. The behaviour of the RFIM on complete graphs is thus quite varied and anomalous. To determine whether this non-universality extends to other lattices, we have analysed the zero-temperature RFIM on a *Bethe* lattice for the stretched exponential and power law distributions of disorder. We prove that *the Bethe lattice is universal*, provided the transition is second order, even in the limit of large co-ordination. This is surprising since in this limit the Bethe lattice usually approaches the mean-field limit.

We also extend the results outlined above to the non-equilibrium case. Ground state calculations of hysteresis and Barkhausen noise in the RFIM have demonstrated that the spin avalanches are controlled by the equilibrium RFIM critical point [10, 11]. It is thus not surprising, and we confirm, that the magnetization jump in the hysteresis loop is non-universal for the stretched exponential disorder distribution. The integrated avalanche distribution also has a non-universal exponent due to the non-universality of the order parameter. But the 'differential' mean-field avalanche exponent is universal even in cases where the order parameter exponent is not. In contrast, as expected from the equilibrium results, the Bethe lattice exhibits universal non-equilibrium critical behaviour.

The Hamiltonian of the RFIM is

$$\mathcal{H} = -\sum_{ij} J_{ij} S_i S_j - \sum_i (H + h_i) S_i, \qquad (1)$$

where the exchange is ferromagnetic $(J_{ij} > 0)$ and the fields h_i are random and uncorrelated. In the non-equilibrium problem we sweep the applied uniform field, H, from $-\infty$ to ∞ and monitor the magnetization at a fixed $J_{ij} = J$ and for a fixed disorder configuration $\{h_i\}$. This model has been proposed as a model for Barkhausen noise by Dahmen *et al* [10]. The local effective field responsible for a spin-flip is

$$h_i^{eff} = J \sum_{j \neq i} S_j + h_i + H. \qquad (2)$$

The condition for a spin to flip is that $h_i^{eff} > 0$. The random fields are drawn from a specified distribution $\rho(h)$. To test universality, we use the following distributions which are defined on the interval $-\delta h \leqslant h \leqslant \delta h$:

$$\rho_1(h) = \frac{y+1}{2y\delta h}\left[1 - \left(\frac{|h|}{\delta h}\right)^y\right] \qquad 0 < y < \infty \qquad (3)$$

and

$$\rho_2(h) = \frac{y+1}{2\delta h}\left(\frac{|h|}{\delta h}\right)^y \qquad -1 < y < \infty. \qquad (4)$$

We have shown that ρ_1, which is the low field expansion of a stretched exponential disorder distribution, leads to non-universality in the ground state of the equilibrium mean-field RFIM [6]. Here we extend that result to the non-equilibrium case. We then show that the distribution ρ_2 destroys the RFIM phase transition in the case of complete graphs (for $-1 < y < 0$), but not in the case of trees.

First we discuss the behaviour of the ground state of the zero-temperature, mean-field RFIM. The magnetization is given by

$$m = -\int_{-\infty}^{h_c(m)} \rho(h)\,\mathrm{d}h + \int_{h_c(m)}^{\infty} \rho(h)\,\mathrm{d}h \tag{5}$$

where $h_c(m) = -Jm - H$. The energy at a given magnetization is

$$E(m) = \frac{Jm^2}{2} - \int_{-\infty}^{\infty} |h|\rho(h)\,\mathrm{d}h + 2\int_0^{h_c} h\rho(h)\,\mathrm{d}h. \tag{6}$$

Extremizing with respect to the order parameter, m, yields the ground state mean-field equation,

$$m_e = 2\int_0^{Jm_e+H} \rho(h)\,\mathrm{d}h. \tag{7}$$

The non-equilibrium critical points are found from the susceptibility $\chi = \partial m/\partial H$, which from (7) is given by

$$\chi = \frac{2\rho(Jm+H)}{1-2J\rho(Jm+H)}. \tag{8}$$

The avalanche distribution, $d(s,t)$, that gives the probability of finding an avalanche of size s at parameter value t, is found using a Poisson statistics argument [10], which yields

$$d(s,t) \sim s^{-\tau}\mathrm{e}^{-t^2 s} = s^{-\tau} g(s^{\sigma} t), \tag{9}$$

where $g(x)$ is a scaling function and $t = 1-2J\rho(Jm_e+H)$. Experimentally, it is more natural to make a histogram of all avalanches up to the critical applied field at which the magnetization changes sign. This 'integrated' distribution behaves as

$$D(s,\delta h) = s^{-\tau-\sigma\beta\delta} g(s^{\sigma} r) \tag{10}$$

where $r = |\delta h - \delta h_c|$. For a Gaussian distribution of disorder, $\beta = 1/2$, $\sigma = 1/2$ and $\tau = 3/2$. We have shown, however, that in the ground state for the distribution (3), the equilibrium order parameter exponent, $\beta = 1/y$. In contrast it is evident from equation (9) that the exponents σ and τ are universal. The non-universality in non-equilibrium behaviour arises in the magnetization jump and in the shape of the non-equilibrium phase boundary, as we now demonstrate. Consider the distribution (3). Integrating (7) yields the mean-field equations,

$$m = \frac{y+1}{y}(\overline{J}m + \overline{H}) - \frac{1}{y}|\overline{J}m + \overline{H}|^{y+1} \tag{11}$$

for $Jm + H > 0$, and

$$m = \frac{y+1}{y}(\overline{J}m + \overline{H}) + \frac{1}{y}|\overline{J}m + \overline{H}|^{y+1} \tag{12}$$

for $Jm + H < 0$. Here we have defined $\overline{J} = J/\delta h$, $\overline{H} = H/\delta h$. Setting $\overline{H} = 0$ in either (11) or (12) yields the equilibrium magnetization [6],

$$m_{eq} = \frac{1}{\overline{J}}[y+1]^{1/y}\left[1 - \frac{y}{\overline{J}(y+1)}\right]^{1/y}. \tag{13}$$

At the critical point, the magnetization scales with the magnetic field as $m_e(r=0,H) \sim H^{1/\delta}$. From equation (12) it is evident that $\delta = y+1$. The susceptibility $\chi = \partial m/\partial\overline{H}$ diverges when the barrier between the two local magnetization minima of the ground state energy ceases to exist. From (8), we have

$$\chi = \frac{(y+1)[1-(\overline{J}m+\overline{H})^y]}{y-(y+1)\overline{J}[1-(\overline{J}m+\overline{H})^y]} \tag{14}$$

and the critical condition

$$y = (y+1)\overline{J}[1-(\overline{J}m_{neq}+\overline{H_c})^y]. \tag{15}$$

This equation has the simple solution

$$x_c = \overline{J}m_{neq} + \overline{H_c} = \left[1 - \frac{y}{\overline{J}(y+1)}\right]^{1/y}. \tag{16}$$

Substituting (16) into (11), we find that the non-equilibrium magnetization jump is positive and has the value

$$m_{neq} = \left[1 + \frac{1}{\overline{J}(y+1)}\right]\left[1 - \frac{y}{\overline{J}(y+1)}\right]^{1/y}, \tag{17}$$

for $H \to H_c^+$. Substituting this into (16), the critical field is found to be

$$H_c = -J\left[1 - \frac{y\delta h}{J(y+1)}\right]^{1+1/y}. \tag{18}$$

This negative critical field is expected when starting with the positive magnetized state. By symmetry, the negative magnetization solution is at $-H_c$ (figure 1(a)). The value of the magnetization at that point is $-m_{neq}$. Note that $|m_{neq}|$ is *not* the size of the magnetization jump in the hysteresis loop. The jump in magnetization in the hysteresis loop is $\delta m_{hyst} = |m_{neq}| + m(|H_c|)$ (figure 1(b)), where $m(|H_c|)$ is found by solving equation (12). The critical exponent associated with the jump in magnetization is determined by the behaviour of the distribution $\rho(h)$ at small fields, so that the critical exponents found here apply to distributions of the form $\rho(h) = \exp(-(|h|/H)^y)$. For $y < 1$ these are the stretched exponential distributions ubiquitous in glasses, while for $y > 2$ they are more concentrated near the origin.

Now we briefly consider the distribution $\rho_2(h)$ given in equation (4). For $y > 0$ this distribution is bimodal and it is easy to confirm the conclusion of Aharony [3] that the transition is first order. However the cases $-1 < y < 0$ are more interesting. In these cases the disorder is dominated by small random fields, as the distribution is singular at the origin. It is easy to carry out the mean-field calculation (7) with the result

$$m_{eq} = \left(\frac{\delta h}{J}\right)^{1+1/y} \qquad \delta h > J. \tag{19}$$

By comparing the energies of $E(m=0)$, $E(m=1)$ and $E(m_{eq})$ (using equation (6)), we find that for $\delta h < J$ the ground state is fully magnetized, while for $\delta h > J$ the ground state has magnetization (19). The interesting feature of the result (19) is that *there is no phase transition* at finite δh, and the system is always ordered. The disorder distribution (4) thus destroys the ground state phase transition, due to the large number of small random fields.

Now we determine whether the non-universal results found above for the mean-field theory extend to the ground state of the RFIM on a Cayley tree. The coordination number of a tree is taken to be z, while the probability that a spin is up is P_+ and the probability that a spin is down is P_-. The probability that a spin is up at level l can be written in terms of the probabilities at the level which is one lower down in the tree; this yields [12, 13]

$$P_+(l) = \sum_{g=0}^{\alpha} \binom{\alpha}{g} P_+^g(l-1)P_-^{\alpha-g}(l-1)a_+(\alpha, g) \tag{20}$$

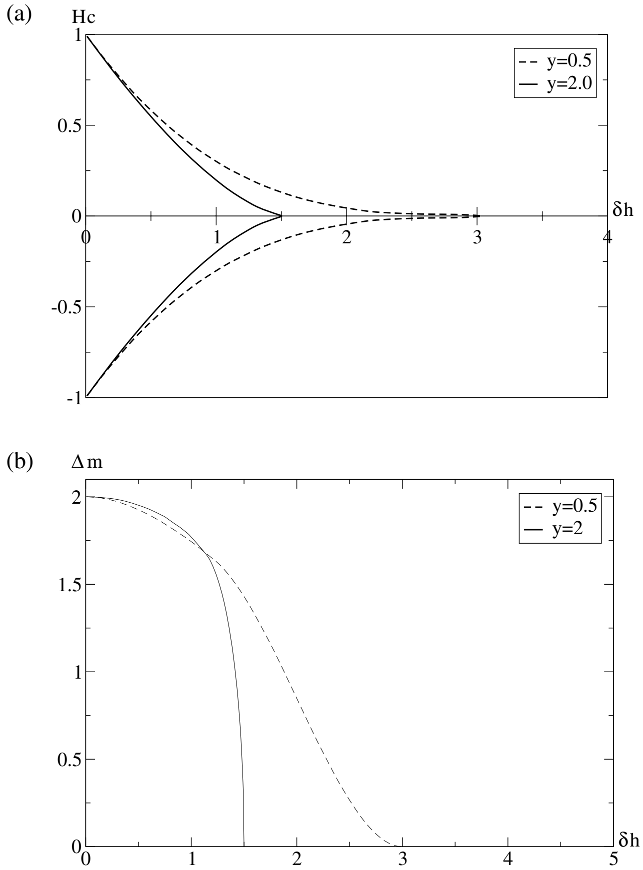

Figure 1. (a) The phase diagram of the non-equilibrium RFIM MFT using the disorder distribution (3), and (b) the magnetization jump at the phase boundary. In these figures, we took the exchange constant $J = 1$. The dotted curve is for $y = 0.5$ while the solid curve is for $y = 2$. Note that at the equilibrium critical disorder, δh_c, the hysteresis loop disappears.

where $a_+(\alpha, g)$ is the probability that the local effective field is positive when g neighbours are up. If we know the distribution $\rho(h_i)$ we can compute $a_+(\alpha, g)$. Analysing the equilibrium behaviour, we have

$$a_+^{eq}(\alpha, g) = \int_{(\alpha-2g)J-H}^{\infty} \rho(h)\, \mathrm{d}h. \tag{21}$$

The equilibrium Cayley tree model has been extended to the non-equilibrium case by considering a growth problem in which the spin *above* the currently considered level in the tree is pinned in the down position [13,14]. This models the growth of a domain. The formalism is the same as in equation (20), with the modification that

$$a_+^{neq}(\alpha, g) = a_+^{eq}(z, g). \tag{22}$$

From this equality and the form (20) it is easy to derive all of the non-equilibrium results from the equilibrium results found using equations (20) and (21). To find the hysteresis curve on a Cayley tree, we just shift the equilibrium magnetization as a function of field: by $H \to H - J$

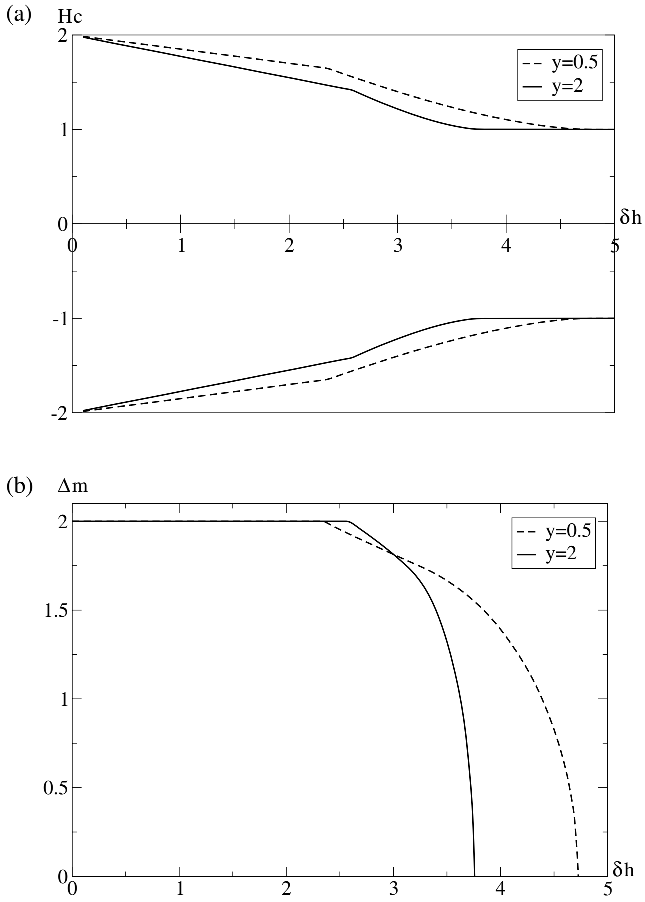

Figure 2. (a): the phase diagram for the non-equilibrium RFIM on a Cayley tree with coordination number $\alpha = 3$ using the distribution (3) and taking the exchange constant $J = 1$. The dotted curve is for $y = 0.5$ and solid curve is for $y = 2$. The initial, linear part, of the phase boundary is due to the finite cut-off of the distribution (3). There is a discontinuity in slope of $H_c(\delta h)$ at the equilibrium critical disorder δh_c. (b): the magnetization jump for the RFIM on Cayley trees for $z = 4$ and the distribution (3), with the exchange constant $J = 1$. The dotted curve is for $y = 0.5$ while the solid curve is for $y = 2$. In both cases we find the same critical exponent, for example $\beta = 1/2$. In contrast, the mean-field result is $\beta = 1/y$.

when sweeping from large positive fields and by $H \to H + J$ when sweeping from large negative fields (figure 2(a)). The behaviour is evident in previous numerical work, but does not seem to have been noticed before.

By direct iteration of the recurrence relation (20) we show that a stable steady state solution, $P_+^* = 1 - P_-^*$, exists. It is easy to solve equation (20) in the steady state limit, at least for small values of α. For $\alpha = 1, 2$, Cayley trees have no ordered state for any finite δh, for the disorder distribution (3). But for $\alpha = 3$ a ferromagnetic state does exist for a range of disorder. As we see from equation (20), the $\alpha = 3$ case leads to a polynomial of order 3 which can be simplified to

$$\frac{m}{4}[m^2(1 - 3b + a) - 1 + 3a + 3b] = 0 \tag{23}$$

where $m = 2(P_+^* - 1/2)$, $a = a_+^{eq}(3, 0)$ and $b = a_+^{eq}(3, 1)$. Equation (23) has the following solutions:

$$m = 0 \qquad \text{and} \qquad m = \pm\left(\frac{3a + 3b - 1}{3b - 1 - a}\right)^{1/2}. \tag{24}$$

These solutions apply for any disorder distribution. For the distribution $\rho_1(h)$, performing the integrals yields

$$m = \left[\frac{4y - 12(y+1)\overline{J} + 3(3^{y+1} + 1)\overline{J}^{y+1}}{3(1 - 3^y)\overline{J}^{y+1}}\right]^{1/2}. \tag{25}$$

We can now expand the magnetization around the critical point, J_c, $\overline{J} = \overline{J}_c - \epsilon$. We find

$$m \sim \left[\left(-12(y+1) + 3\binom{y+1}{y}(1 + 3^{y+1})\overline{J}^y\right)\epsilon\right]^{1/2}. \tag{26}$$

Thus $m \sim \epsilon^{1/2}$ for any y, so that $\beta = 1/2$ is universal (figure 2(b)). Since the non-equilibrium behaviour on trees is related to that of the equilibrium behaviour in such a simple manner, this universality extends to the hysteresis and avalanche exponents. It is easy to confirm numerically that the behaviour extends to large values of the branch co-ordination number α. Moreover by doing an expansion of (20) using $P_+ = 1/2 + m$, it is possible to show analytically that only the first- and third-order terms in m exist, regardless of the value of y in the disorder distribution (3). This confirms that for this distribution, the behaviour is universal for all coordination numbers.

For the distribution $\rho_2(h)$ and $\alpha = 3$ we get from equation (24)

$$m = \left[\frac{4 - 3(3^{y+1} + 1)\overline{J}^{y+1}}{3(3^y - 1)\overline{J}^{y+1}}\right]^{1/2}. \tag{27}$$

Just as we have done before we can expand m around the critical point, $\overline{J} = \overline{J_c} - \epsilon$:

$$m \sim \left[\binom{y+1}{y}(3^{y+1} + 1)\overline{J_c}^{y}\epsilon\right]^{1/2}. \tag{28}$$

Thus for ρ_2, $\beta = 1/2$ is a universal exponent, too.

In summary, on complete graphs (i.e. in mean-field theory) the RFIM at $T = 0$ is non-universal. In particular, the stretched exponential disorder distribution leads to a non-universal order parameter exponent and non-universal integrated avalanche exponent. In addition, the power law distribution has a regime in which a predominance of small random fields destroys the transition and the RFIM always has a finite magnetization. In contrast the Cayley tree does not show either of these behaviours. Even in the limit of large coordination it is universal, with the usual mean-field order-parameter exponent $1/2$. We have carried out some preliminary numerical studies of the behaviour in three dimensions (with short range interactions) and find that the power law distribution of random fields does *not* destroy the transition. Moreover, the exceedingly small value of β in three dimensions renders any non-universality in β a moot point. However the behaviour in dimensions higher than three, or for longer range interactions in three dimensions, could be more interesting. Finally, even for short range interactions in three dimensions, there have been suggestions of non-universality in the finite size scaling behaviour [9]. It is unclear, as yet, whether that behaviour is related to the non-universality seen here.

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